Gaining diagnostic teaching skills:
helping students learn from mistakes and misconceptions

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Summary

This package provides material for supporting teachers in developing diagnostic teaching skills. It includes diagnostic assessment and instructional material for helping students who have difficulties in mathematics to identify and deal with their mistakes and misconceptions. These tools will enable teachers to help their students use common mistakes and misconceptions constructively in the classroom to promote more effective learning of mathematical concepts.

These materials, and the detailed suggestions for their use, focus on common difficulties within a number of mathematical topics. They support lessons that helps student identify their own and others' errors and, through discussion and reflection, understand and correct the misconceptions that underlie them.

Purpose

To equip teachers to use student mistakes and misconceptions to teach more effectively for long term learning.

Tool Description

The elements in this set of materials include:

- diagnostic tests
- guidance for scoring and interpreting responses
- samples of student work
- teaching materials
- outlines for professional development sessions

Topics The materials focus on common student difficulties in a number of the key mathematical topics that trouble students from late elementary grades through early high school:

- *Decimals and Fractions*
- *Number Operations*
- *Functions and Graphs*
- *Algebra*
- *Geometry*

along with a *General introduction* to the principles of the approach.

Users These resources are designed to be used in a variety of different modes by:

- teachers in the classroom
- teachers in an extra period (double period) of mathematics
- teachers in after school programs
- teachers in summer school programs

- intervention specialists
- tutors, working with small groups of students in 'extra time'
- professional development leaders helping teachers to improve their skills in this area

Background

Research shows that students may appear successful in following the normal curriculum yet still have a number of serious misconceptions which cause persistent errors. This happens mainly because in the normal course of instruction, skills are taught and then practiced on problems for which these skills are the ones needed. The students are not often required to recognize a fresh, unfamiliar situation and select an appropriate method from their wider range of knowledge. The body of knowledge they build up is thus superficial. For example, students acquire from their elementary school experience the awareness that multiplying a number makes it bigger. This may or may not be made explicit. Later, when decimal numbers come into use, the fact that this principle does not apply when the multiplier is less than 1 may again not be made explicit, and this leads to errors in choosing correct operations.

Routine practice on standard problems does little to help students overcome common mistakes and misconceptions in mathematics. This is particularly true when teachers try to avoid difficulties arising by beginning each lesson with explanations and demonstrations and following this with carefully graded questions.

Improving student learning depends on establishing a more effective *classroom contract* – the usually unspoken agreement between teachers and students as to the roles each of them will play. Traditionally, the teacher with the textbook explains and demonstrates, while the students imitate; if the student makes mistakes the teacher explains again. This procedure is not effective in preventing the development of misconceptions or in removing pre-existing ones. Diagnostic teaching, and much else including non-routine problem solving, depends on the student taking more responsibility for their own understanding, being willing and able to articulate their own lines of thought and to discuss them in class and with their peers.

Thus, effective teaching requires that common mistakes and misconceptions are systematically confronted and explored within the classroom. The lessons described here typically begin with a challenge that exposes students' existing thinking. Students are then confronted with cognitive conflicts that challenge these ways of thinking. New ideas are constructed through reflective discussion. This approach is challenging but research shows that it develops connected, long-term learning.

Design principles

The approach is based on the following principles:

- Lessons focus on known, specific difficulties. Rather than posing many questions in one session, it is more effective to focus on a challenging situation or context and encourage a variety of interpretations to emerge, so that students can compare and evaluate them.
- Questions or stimuli are posed or juxtaposed in ways that create a tension or conflict that needs resolving. Contradictions arising from conflicting methods or opinions create awareness that something needs to be reconsidered, and understandings clarified.
- Activities provide opportunities for meaningful feedback to the student on his or her interpretations. This does not mean providing superficial information, such as the number of correct or incorrect answers. Feedback is provided by students using and

comparing results obtained from alternative methods. This usually involves some form of small group discussion.

- Lessons include time for whole class discussion in which new ideas and concepts are allowed to emerge. This can be a complex business and requires non-judgmental sensitivity on the part of the teacher so that students are encouraged to share tentative ideas in a non-threatening environment.
- Opportunities are provided for students to 'consolidate' what has been learned through the application of newly constructed concepts.
- Each session tackles both content and process issues, as outlined in the NCTM Standards. In particular, all sessions have a particular focus on developing communication, connections and representation.
- The Standards also embrace a third dimension – principles that are intended to provide teachers with guidance. These are the principles of *Equity* (high expectations), *Curriculum* (coherence), *Teaching* (building on what students already know), *Learning* (developing meaning through students' own constructions), *Assessment* (focus on the important) and the use of *Technology*. We take careful note of these principles and their application to all mathematics teaching.

As an illustration of the materials in these Diagnostic Teaching tools, we give the outline for the General Introductory course below.

The materials

These include:

- diagnostic tests
- guidance for scoring and interpreting responses
- samples of student work
- teaching materials
- outlines for professional development sessions

These aspects are explained and illustrated in the Introduction to the materials that follows.

Using the tool

Having decided to develop diagnostic teaching:

- Get a colleague to work on it with you. Each chooses a suitable class, for which one of the topics treated in the tool is appropriate.
- Get together the materials given in the tool. Read the references given here. Plan together what you will do.
- Get your colleague to observe your lessons and discuss them with you – and vice versa. Try with another class, adapting your approach in the light of the observations.
- As you gain confidence, try some of the other topics.

The package may be used in a number of different ways, for different purposes. They are described on the following pages. The three main modes are

1. a course for developing diagnostic teaching skills, either by individual teachers or groups or as a leader-run professional development course;
2. using diagnostic assessment to guide instruction;
3. material to support instructional interventions.

1. Developing diagnostic teaching skills

The teacher is a facilitator of the diagnostic teaching process, rather than the expected source of all understanding. These materials support a sequence of professional development sessions that enable teachers to develop the additional classroom skills involved (as well as deepening their own understanding of mathematics).

This can be of great long-term value, enabling teachers to develop their classroom skills to give much more effective teaching across the whole program. Outlines and resource material for six professional development courses are provided. These can be followed by individual teachers (or preferably by a group) working on their own. However, it is likely to be much more effective if taken as a professional development course, run by a leader.

In either case, it is recommended to start with the ***General Introductory Course***.

This preferably covers a period of five weeks, and includes:

a. an initial meeting introducing the course program and materials (about 1 hour);
b. a week during which the teachers give a diagnostic test to one of their classes, and score and analyze the students' responses;
c. two course sessions of 1.5 hours each, in which teachers discuss their students' work, read relevant research, and prepare diagnostic lessons together;
d. a two-week period in which teachers try the prepared lessons in their classrooms;
e. a follow-up meeting of about 1.5 hours for discussion of results, teaching, problems met and how solved, response of students, plans for further development.

A fuller outline of this course is given below.

Topic-specific courses. The remaining five courses follow a similar pattern. Each introduces the teachers to the diagnostic method in the context of one of the five key topics listed above.

2. Diagnostic assessment to guide instruction

It is important that sufficiently probing assessments be used to reveal the state of the students' underlying understanding, so that appropriate instruction may be given.

A few questions from the *test on understanding decimal operations*, and a typical example of misunderstanding, are shown in *Example 1*. Here, Damien gives $10 \div 0.5 = 2$, $0.2 \times 0.4 = 0.8$, says the answer to $26.12 \div 0.286$ is bigger than 13 and smaller than 26, and that$19 \div 76$ is the same as How many 19s go into 76.

This diagnostic testing would be suitable at the beginning of the year in, say, grades 5, 6, 7, 8, and 9, when it would give a general indication of the extent of students' problems. However, giving all five tests together is far from ideal. More usefully, when one of these key topics is about to appear in the instructional program, give the relevant *diagnostic test* towards the end of the previous unit. This will provide valuable information on what aspects need particular attention, and the test items themselves can be used as starting points for discussion of the conceptual problems shown up.

Appropriate *teaching material* for about 5 lessons is provided in each of the five topic-based packages.

An example of one activity on choosing operations is shown in *Example 2*. It involves placing each one of a set of eight cards containing word problems on a pile labelled by the operation needed to give the answer.

3. Instructional interventions

Where groups of students are identified as being weak, or below the performance level for their grade, either generally or in specific areas, the diagnostic tests can pinpoint their particular gaps in understanding, and the teaching material supplied in these packages may be used to provide necessary interventions. These may take the form of lessons of normal length, or in short classroom episodes of about 15 minutes per day, or in an after-school program, according to needs. An example of this teaching material is in Appendix 2.

Evaluative evidence

There is *research evidence* that shows, across a range of topics, that the diagnostic teaching approach leads to much better long term learning than standard “positive only” methods, which avoid analyzing and understanding misconceptions.

This is summarized in the strategy <u>*Gain diagnostic teaching skills*</u> and in

Askew, M. and D. Wiliam (1995). *Recent Research in Mathematics Education 5-16.* London, HMSO and covered more fully in the references given below.

Strengths

- Helps students develop a *robust cognitive structure* that enable long-term learning;
- Students take more responsibility for their own learning – a key aspect of higher performance, across mathematics and the wider curriculum.

Likely challenges

- Teacher-student and student-student discussion of the mathematics is central to this activity; you need to handle this in a facilitative non-directive way, so that students feel responsible for their own learning.
- This will present a new challenge for many teachers, by extending the usual range of teaching activities.

There are rewards in the satisfaction of meeting students’ real problems, and of achieving robust learning for your students.

Availability

These materials are available, either free on the web or from Shell Centre Publications, as follows:

General Introductory Course - free

Topic-specific courses:

- *Decimals and Fractions - free*
- *Number Operations - at www.mathshell.com*
- *Functions and Graphs*
- *Algebra*
- *Geometry*

References

Askew, M. and D. Wiliam (1995). *Recent Research in Mathematics Education 5-16.* London, HMSO.

A theoretical background and reports of some teaching experiments can be found in

Bell, A. 1993a. Principles for the design of teaching. *Educational Studies in Mathematics 24,* 5-34.

Bell, A, 1993b Some experiments in diagnostic teaching, *Educational Studies in Mathematics 24,* 115-137.

Additional topic-specific references are given in the lesson resources.

Some exemplars from the materials

The following examples give some thing of the flavor of this set of tools

Introduction

Research shows that the *diagnostic teaching* approach, by confronting and analyzing errors, leads to more robust long-term learning. This package provides training for teachers in developing diagnostic teaching skills, and includes a supporting collection of material. Thus it will enable teachers and tutors to help all students, including those who normally have difficulties, to achieve effective learning, by dealing directly with their mistakes and misconceptions,

Topics

These materials take as their context the common student difficulties in these key mathematical topics:

- *Decimals and Fractions* – Understanding their meaning in all common representations, symbols, diagrams, number line, and in practical situations; also behavior of decimal numbers when multiplied or divided by 10, or by other numbers, greater or less than 1.
- *Number Operations* – More extensive study of the way numbers are changed by multiplying or dividing by numbers greater and less than 1, awareness of non-commutativity of subtraction and division, choice of correct operation in practical problems, especially when the size of the numbers causes distraction, ability to give a rough estimate of the result of a calculation, and to relate this to the correct operation, for example on a calculator, conversion between units, speed and distance problems, making up problems to fit given calculations.
- *Functions and Graphs* - Plotting points, using equation of a line, using and interpreting graphs of practical situations – scatter graphs, distance or speed and time graphs, relating graphs to stories and to practical situations, identifying meaning of points of intersection of two graphs.
- *Algebra* – Expressing patterns and situations by simple algebraic expressions, simple manipulation, forming equations from problems and solving them, expressing functions, interpreting effect of changing values of variables on the value of the expression.
- *Geometry* – angles, polygons, area, perimeter, volume, enlargement, scale drawing.

Example 1

Damien: Multiplication and Division

This shows how well-designed diagnostic questions can reveal important aspects of a student's understanding

1. Do these in your head and write down your answers as decimals:

(a) $4 \div 20 =$ 5.0

(b) $6 \times 0.5 =$ 3.0

(c) $10 \div 0.5 =$ 2

(d) $0.7 \div 0.7 =$ 0.7

(e) $0.2 \times 0.4 =$ 0.8

(f) $60 \div 0.3 =$ 20

(g) $60 \times 0.3 =$ 180

(h) $16 \div 20 =$ ....

5. The answer to $26.12 \div 0.286$ will be....
Ring two correct statements

Bigger than 26 — Smaller than 26

Bigger than 13 — Smaller than 13

Give a **rough estimate** of the answer: 17.00

7.

(a) The boxes contain **six** statements.
Tick every statement that means the same as **85 ÷ 17**

How many 17's go into 85 ? ✓

What fraction of 85 is 17 ?

85)17 ✓ | 17)85 | $\frac{17}{85}$ ✓ | $\frac{85}{17}$

(b) **Tick every** statement that means the same as **19 ÷ 76**

How many 19's go into 76? ✓

What fraction of 76 is 19 ?

76)19 | 19)76 ✓ | $\frac{19}{76}$ ✓ | $\frac{76}{19}$

Example 2

Cards for sorting: Multiplication and Division
Cut out the cards and sort, putting each problem with its correct answer

This is an example of how a well-structured task can expose and practice key concepts

I share 5 liters of lemonade equally among 20 people. How much lemonade does each person get?	I share 20 liters of lemonade equally among 5 people. How much lemonade does each person get?
I pour 20 liters of water into buckets. Each bucket holds 5 liters. How many buckets do I fill?	I pour 5 liters of water into a tank which holds 20 liters. What fraction of the tank does it fill?
I cut a 20 meter rope into 0.5 meter lengths. How many pieces do I get?	I cut a 0.5 meter wire into 20 equal pieces. How long is each piece?
A box holds 20 jars of jam. Each jar weighs 0.5 kg. What weight is in the box?	A full box holds 20kg of potatoes. It is half full. What weight does it contain?
$20 \div 5$	$5 \div 20$
$20 \div 0.5$	$0.5 \div 20$

Outline for the General Introductory Course

This is suitable for middle school and high school teachers, preferably working together as a group. It touches on fractions and decimals; multiplication and division; area and perimeter; algebraic notation.

1. Where are we starting from with ourselves?
(30 mins)

Look at the sheet **'Beliefs about teaching and learning math'** in the handbook. This consists of a number of cards that teachers should cut up and discuss in groups. They record their main thoughts and feelings as they do this so that they might report back.

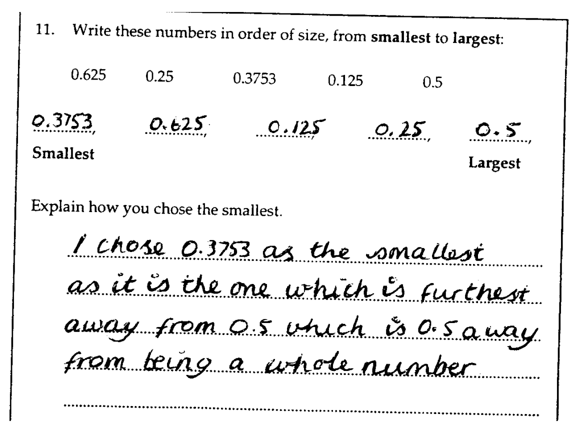
This activity both models one of the two activity types we shall be using in the session and it also provokes teachers to reflect on the reasons why they work in the ways that they do.

Math is best learned through practice.	Math is best learned through discussion.
Students learn math best when they work on their own.	Students learn math best when they work collaboratively.
Math is a network of ideas.	Math is a hierarchical subject.
It is best to begin teaching math with easy problems, working gradually up to harder ones, otherwise students make mistakes and lose confidence.	It is best to begin teaching math with complex problems, or students won't appreciate the need for it.

2. Where are we starting from with students? Looking at students' work (30 minutes)

The group begins by looking at a selection of students' work.

These are responses to diagnostic questions in four different topics; Fractions and decimals, Multiplication and division, Area and perimeter, Algebraic notation. People are asked to summarize the errors that have been made and the thinking that may have led to these errors. This activity is intended to encourage teachers to see that errors may be due to deep-rooted misconceptions that should be exposed and discussed in classrooms.


11. Write these numbers in order of size, from **smallest** to **largest**:

0.625 0.25 0.3753 0.125 0.5

0.3753 0.625 0.125 0.25 0.5

Smallest **Largest**

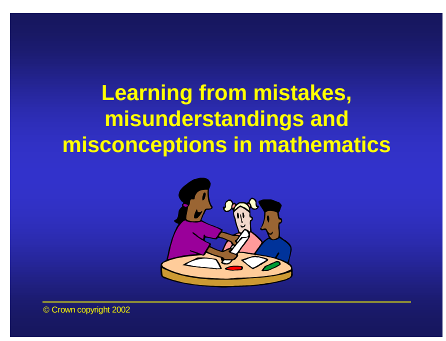
Explain how you chose the smallest.

I chose 0.3753 as the smallest as it is the one which is furthest away from 0.5 which is 0.5 away from being a whole number

3. Plenary discussion (30 minutes)

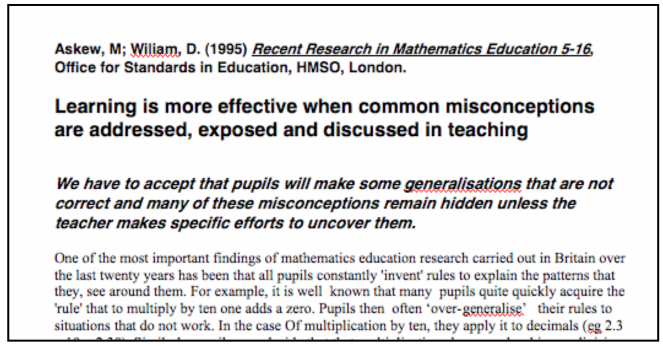
Using the *PowerPoint presentation*,

Display the student work and ask for comments. Do you recognize any of these errors from their own experiences? What other similar types of error might arise?



Lead into a general discussion on why students make mistakes, and, in pairs, discuss and suggest further examples of local generalizations that are not always true. In addition describe common approaches to dealing with mistakes and errors. What does research say? (20 minutes)

Issue copies of the one-page research review by *Askew and Wiliam (1995)*. This describes some research that was conducted in Britain over the past 20 years on dealing with student errors and difficulties.


Askew, M; Wiliam, D. (1995) *Recent Research in Mathematics Education 5-16*, Office for Standards in Education, HMSO, London.

Learning is more effective when common misconceptions are addressed, exposed and discussed in teaching

We have to accept that pupils will make some generalisations that are not correct and many of these misconceptions remain hidden unless the teacher makes specific efforts to uncover them.

One of the most important findings of mathematics education research carried out in Britain over the last twenty years has been that all pupils constantly 'invent' rules to explain the patterns that they, see around them. For example, it is well known that many pupils quite quickly acquire the 'rule' that to multiply by ten one adds a zero. Pupils then often 'over-generalise' their rules to situations that do not work. In the case Of multiplication by ten, they apply it to decimals (eg 2.3

- What do people feel about the dilemmas presented in this paper?
- Is it possible to present examples where rules do not work?
- Do people feel that it is counter-productive to teach simpler examples before more complex examples?

A more detailed consideration of how to deal with misconceptions is given in the second paper, 'Dealing with misconceptions in mathematics' by *Malcolm Swan (2001)*. You may like to issue this as further background reading for later consideration.

4. **Developing classroom approaches 1:** (30 minutes)

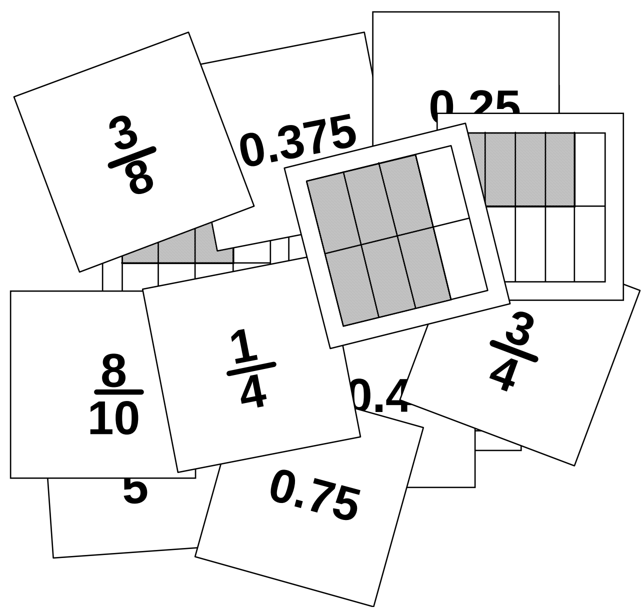
'Collecting together different representations of a concept'

Refer to the sample classroom activities in the workbook. The first type concerns 'collecting together different representations of a concept'.

Allow time to try out the first activity for themselves in groups.

Two examples are provided: Decimals and Fractions and Multiplication and Division.

As people do this, then ask them to reflect on their own thought processes and answer the questions at the foot of page 11 in the workbook.



5. **Developing classroom approaches 2:** (30 minutes)

‘***Always, sometimes and never true’.***

Allow time to try out the second activity types in groups.

Two examples are provided: Area and perimeter and Algebra.

People may then enjoy trying to invent some similar statements of their own to use in their own classroom (in any topic they like).

Draw two rectangles. The one with the greater area will also have the greater perimeter.
If you cut a piece out of a rectangle, you make its area smaller.
If you cut a piece out of a rectangle, you make its perimeter smaller.
A square and a rectangle both have the same perimeter. The square has the greater area.
A square and a rectangle both have the same area. The square has the greater perimeter.

6. **Preparing to teach** (30 minutes)

Each group should now choose one idea and use it to develop a lesson that they can try out during the following week. They should write out the lesson plan in as much detail as possible, including times, good questions to ask, etc.

7. **Into the classroom and reporting back**

Take some of these ideas into the classroom and arrange a follow-up meeting when you can share experiences. Bring along:

- Three pieces of student work – strong, medium and weak;
- Photographs or artifacts to help describe what happened.